

Can large N_c equivalence between supersymmetric Yang-Mills theory and its orbifold projections be valid?

Pavel Kovtun^{*}, Mithat Ünsal[†] and Laurence G. Yaffe[‡]

^{*}*KITP, University of California, Santa Barbara, CA 93106*

[†]*Department of Physics, Boston University, Boston, MA 02215*

[‡]*Department of Physics, University of Washington, Seattle, Washington 98195-1560*

Email: kovtun@kitp.ucsb.edu,
unsal@buphy.bu.edu,
yaffe@phys.washington.edu

ABSTRACT: In previous work, we found that necessary and sufficient conditions for large N_c equivalence between parent and daughter theories, for a wide class of orbifold projections of $U(N_c)$ gauge theories, are just the natural requirements that the discrete symmetry used to define the projection not be spontaneously broken in the parent theory, and the discrete symmetry permuting equivalent gauge group factors not be spontaneously broken in the daughter theory. In this paper, we discuss the application of this result to \mathbb{Z}_k projections of $\mathcal{N}=1$ supersymmetric Yang-Mills theory in four dimensions, as well as various multi-flavor generalizations. \mathbb{Z}_k projections with $k > 2$ yielding chiral gauge theories violate the symmetry realization conditions needed for large N_c equivalence, due to the spontaneous symmetry breaking of discrete chiral symmetry in the parent super-Yang-Mills theory. But for \mathbb{Z}_2 projections, we show that previous assertions of large N_c inequivalence, in infinite volume, between the parent and daughter theories were based on incorrect mappings of vacuum energies, theta angles, or connected correlators between the two theories. With the correct identifications, there is no sign of any inconsistency. A subtle but essential feature of the connection between parent and daughter theories involves multi-valuedness in the mapping of theta parameters from parent to daughter.

KEYWORDS: 1/N Expansion, Spontaneous Symmetry Breaking.

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1. Introduction

Orbifold projection is a technique for constructing “daughter” theories starting from some “parent” theory, by retaining only those fields which are invariant under a chosen discrete symmetry group of the parent theory. In some cases involving orbifold projections of $U(N_c)$ gauge theories, the large N_c limits of parent and daughter theories may coincide [1–5]. More precisely, for a wide class of projections yielding daughter theories with $U(N)^k$ product gauge groups, it has recently been shown, rigorously, that the leading large- N_c dynamics in the sector of the parent theory invariant under the discrete symmetry used to define the projection coincides with the large N_c dynamics in the sector of the daughter theory which is invariant under permutations of equivalent gauge group factors [6]. The ground states of each theory will lie in these symmetry sectors (which we will refer to as *neutral*¹) provided the symmetries

¹In the parent theory, neutral operators (or states) are gauge invariant operators (or states) which are invariant under the chosen projection symmetry. In the daughter theory, neutral operators/states are gauge invariant operators/states which are also invariant under any additional global symmetries, such as permutations of different gauge group factors, which are remnants of gauge symmetries in the parent theory. In string theory literature, non-neutral operators of the daughter theory are referred to as “twisted”.

defining the projection are not spontaneously broken in the parent theory, and the gauge group permutation symmetries are not spontaneously broken in the daughter. These symmetry realizations constraints are both necessary and sufficient for the validity of a non-perturbative equivalence between parent and daughter theories relating the leading large N_c behavior of physical properties such as mass spectra and scattering amplitudes of neutral particles [6].

In this paper, we examine the application of this result to \mathbb{Z}_k projections of $\mathcal{N} = 1$ supersymmetric Yang-Mills theory in four dimensions with gauge group $U(2N)$, or its multi-flavor generalizations. Much of our discussion will focus on the simplest case of a \mathbb{Z}_2 projection of $\mathcal{N} = 1$ super-Yang-Mills, which yields a non-supersymmetric $U(N) \times U(N)$ gauge theory. This example has been discussed previously by a number of authors [3, 7–9]. Gorsky and Shifman [7], and Armoni, Shifman, and Veneziano [9] argued that non-perturbative large N equivalence fails in this case, based on apparent mis-matches in topological susceptibility, and of instanton zero modes when the theory is compactified on T^4 or $R^3 \times S^1$. Tong [8] also considered this example when compactified on $R^3 \times S^1$ and demonstrated, explicitly, that the orbifold equivalence fails when the compactification radius is sufficiently small. Based on this and closely related examples, Ref. [9] suggests that large- N orbifold equivalence may generally fail.

We will show that the apparent discrepancies discussed in Refs. [7, 9] between these parent and daughter theories, formulated on R^4 , are illusory, and result from incorrect mapping of observables between the two theories. In particular, in these papers it was assumed that a valid equivalence would imply that the vacuum energy densities and topological susceptibilities of parent and daughter theories (when supersymmetry is softly broken) would coincide. This is incorrect. As we discuss, in this example the vacuum energy density of the parent theory must be compared with twice the energy density of the daughter, while the topological susceptibility of the parent should be half that of the daughter (because the θ angle of the parent must be identified with twice the θ angle of the daughter theory²).

With the correct mapping between parent and daughter theories, all evidence for any inconsistency of the large- N orbifold equivalence, in the uncompactified theories, disappears. A subtle feature, which is essential for understanding the correspondence between individual vacua, or domain walls, in the two theories, involves multi-valuedness in the mapping of theta parameters between parent and daughter theories. This is discussed in some detail. We argue that there is no sign of any inconsistency with large N equivalence between these two theories, when formulated on R^4 , and no evidence suggesting spontaneous symmetry breaking of the \mathbb{Z}_2 symmetry interchanging the $U(N)$ factors in the daughter theory. However, as

²This is the same relation which results from the mapping of the holomorphic coupling $\tau \equiv 4\pi/g^2 + i\theta/2\pi$ in examples where both parent and daughter are supersymmetric [10]. For a \mathbb{Z}_2 projection, coinciding 't Hooft couplings requires $\tau_{\text{parent}} = 2\tau_{\text{daughter}}$, implying that $\theta_{\text{parent}} = 2\theta_{\text{daughter}}$.

Tong [8] showed quite explicitly, if the theories are compactified on $R^3 \times S^1$ then at sufficiently small radius the symmetry interchanging the $U(N)$ factors in the daughter theory does break spontaneously, thereby violating the necessary conditions for large N orbifold equivalence.

We also examine extensions of these theories involving additional adjoint fermions, and explain why a mis-match in the total number of Goldstone bosons, noted in Ref. [7], is also not evidence for failure of large- N orbifold equivalence in these examples. We find that the number of Goldstone bosons in the relevant neutral symmetry channels, to which the equivalence applies, do match.

We briefly discuss \mathbb{Z}_k projections with $k > 2$, which can yield $U(N)^k$ chiral gauge theories with bifundamental fermions. Large N equivalence between the parent and daughter theories does fail for these projections, due to the spontaneous symmetry breaking of discrete chiral symmetry in the parent theory and in complete accord with the symmetry realization conditions discussed in Ref. [6]. We also comment on the case of \mathbb{Z}_2 projection of $\mathcal{N}=4$ supersymmetric Yang-Mills theory yielding a non-supersymmetric $U(N)^2$ gauge theory with adjoint scalars and bifundamental fermions. This case, which has received considerable previous attention (see, for example, Refs. [11–16]) appears to be an example of large N inequivalence between parent and daughter theories due to spontaneous \mathbb{Z}_2 symmetry breaking in the daughter. However, this does not imply that any analogous symmetry breaking must occur in the $\mathcal{N}=1$ case.

As this paper was (finally) nearing completion, the recent preprint [21] appeared. This work also discusses the consistency of large N orbifold equivalence for the \mathbb{Z}_2 projection of $\mathcal{N}=1$ super-Yang-Mills, but reaches radically different conclusions from ours, namely manifest inconsistency of large N equivalence based on multiple different lines of reasoning. In an addendum to the conclusion of this paper, we briefly examine the assertions of Ref. [21] and find that the multiple apparent inconsistencies discussed in that work can all be traced to incorrect mapping of operators and correlation functions between the two theories. Once the correct mappings are used (as spelled out below and in Ref. [6]), all evidence for failure of large N equivalence in this example (and hence all evidence for spontaneous breaking of the \mathbb{Z}_2 symmetry in the daughter theory) disappears.

2. \mathbb{Z}_2 projection of $\mathcal{N}=1$ supersymmetric Yang-Mills theory

As has been recognized previously [3, 7–9], an interesting example of orbifold projection is provided by the \mathbb{Z}_2 projection of $\mathcal{N}=1$ supersymmetric Yang-Mills theory in four dimensions. The parent theory is a $U(2N)$ gauge theory with a massless Weyl fermion in the adjoint representation.³ The daughter theory is a $U(N) \times U(N)$ gauge theory with a bifundamental

³The difference between $SU(2N)$ and $U(2N)$ gauge groups is irrelevant in the large N limit.

Dirac fermion. Both parent and daughter theories are asymptotically free, confining gauge theories with massless fermion fields. The parent theory is supersymmetric, but the daughter theory is not. Therefore the equivalence, if true, would dictate exact relations in the large N limit between mass spectra, correlators, and scattering amplitudes of a supersymmetric parent theory and its non-supersymmetric daughter [3].⁴

The authors of Refs. [7, 9] argued that the topological susceptibilities of the two theories, in infinite volume, do not match, thus invalidating the equivalence. More precisely, they considered the theories with a small fermion mass turned on, and found that if the vacuum energies of the two theories coincide at $\theta = 0$, then the curvatures of the vacuum energies with respect to θ do not match. We wish to reexamine this comparison between parent and daughter theories. The action density of the parent theory may be written as

$$\mathcal{L}^{(p)} = -\frac{1}{2g_p^2} \text{tr} F_{\mu\nu} F^{\mu\nu} + \frac{\theta_p}{16\pi^2} \text{tr} F_{\mu\nu} \tilde{F}^{\mu\nu} + \frac{1}{g_p^2} \text{tr} [\bar{\lambda}_{\dot{\alpha}} (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} D_\mu \lambda_\alpha + \frac{1}{2} m_p (\lambda^\alpha \lambda_\alpha + \bar{\lambda}_{\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}})] . \quad (2.1)$$

The traces are over $2N \times 2N$ matrices with $F_{\mu\nu} \equiv F_{\mu\nu}^a t^a / (2i)$, and $\lambda_\alpha \equiv \lambda_\alpha^a t^a / (2i)$ an adjoint representation Weyl fermion. With an anomalous chiral rotation, the theta parameter may be moved into the fermion mass term. In the massless limit, the non-anomalous discrete \mathbb{Z}_{4N} chiral symmetry of the parent theory is spontaneously broken to its \mathbb{Z}_2 subgroup by the vacuum fermion condensate, giving rise to $2N$ distinct vacua which are related to each other by shifts in θ_p by multiples of 2π . The unbroken \mathbb{Z}_2 symmetry is just $\lambda \rightarrow -\lambda$, corresponding to conservation of fermion number modulo two. When a small mass is added, the resulting vacuum energy density is

$$\mathcal{E}^{(p)} = \frac{m_p}{2g_p^2} \left[\langle \text{tr} \lambda \lambda \rangle e^{i\theta_p/(2N)} + \langle \text{tr} \bar{\lambda} \bar{\lambda} \rangle e^{-i\theta_p/(2N)} \right] + O(m_p^2) , \quad (2.2)$$

where the phase of $\langle \text{tr} \lambda \lambda \rangle$ is an integer multiple of π/N ; the particular value depends on which of the $2N$ vacua of the massless theory is under consideration. Changing $\theta_p \rightarrow \theta_p + 2\pi$ is the same as moving from one vacuum state to the next. The magnitude of $\langle \text{tr} \lambda \lambda \rangle$ is $O(N\Lambda^3)$, where Λ is the conventional non-perturbative scale of the theory [2, 17]. Therefore, the energy density (2.2) is $O(N^2)$ [since g_p^2 is $O(1/N)$ at fixed 't Hooft coupling] as required for the leading large N behavior in a $U(2N)$ gauge theory.

The \mathbb{Z}_2 orbifold projection eliminates all degrees of freedom except those invariant under the combination of a global gauge transformation by $\gamma \equiv \text{diag}(1_N, -1_N)$ (with 1_N denoting

⁴For example, unbroken supersymmetry in the parent theory implies that massive scalar particles in the smallest supermultiplet must be degenerate in mass with both a pseudoscalar and a spin 1/2 fermion. Only bosonic particles lie in the neutral sector of the parent theory. Hence, the mass spectrum of the daughter theory, in the $N \rightarrow \infty$ limit, must also exhibit parity doubled scalars and pseudoscalars, provided the \mathbb{Z}_2 symmetry interchanging gauge group factors is not spontaneously broken.

an $N \times N$ identity matrix), combined with a fermion sign flip, $\lambda \rightarrow -\lambda$. This forces the gauge field A_μ to be block-diagonal, reducing it from a $U(2N)$ gauge field to two independent $U(N)$ gauge fields, whose field strengths we will denote as $F_{\mu\nu}^1$ and $F_{\mu\nu}^2$. The extra sign change for the fermion means that λ becomes block off-diagonal, so that the daughter theory contains two Weyl fermions transforming as bifundamentals under $U(N) \times U(N)$, which we will denote as λ^1 and λ^2 . (See Refs. [7, 9] for a more detailed explanation.) Under this projection,

$$\text{tr } FF \rightarrow \text{tr } (F^1 F^1 + F^2 F^2), \quad (2.3)$$

$$\text{tr } \lambda\lambda \rightarrow \text{tr } (\lambda^1 \lambda^2 + \lambda^2 \lambda^1) = 2 \text{tr } \lambda^1 \lambda^2, \quad (2.4)$$

where traces on the left are over $2N \times 2N$ matrices, while those on the right are $N \times N$. The action density of the daughter theory has the form

$$\mathcal{L}^{(d)} = \sum_{j=1}^2 \left\{ -\frac{1}{2g_d^2} \text{tr } F^j F^j + \frac{\theta_d}{16\pi^2} \text{tr } F^j \tilde{F}^j + \frac{1}{g_d^2} \text{tr } \bar{\lambda}^j \bar{\sigma}^\mu D_\mu \lambda^j \right\} + \frac{m_d}{g_d^2} \text{tr } (\lambda^1 \lambda^2 + \bar{\lambda}^2 \bar{\lambda}^1), \quad (2.5)$$

where the fermion covariant derivative includes the appropriate coupling to both $U(N)$ gauge fields. This theory has a non-anomalous discrete \mathbb{Z}_{2N} chiral symmetry, which is also expected to break spontaneously due to the formation of a fermion bilinear condensate, leaving an unbroken \mathbb{Z}_2 subgroup (corresponding to fermion number modulo two). In addition, the daughter theory has a \mathbb{Z}_2 “theory space” symmetry which interchanges the two $U(N)$ gauge groups (*i.e.*, $F^1 \leftrightarrow F^2$, $\lambda^1 \leftrightarrow \lambda^2$). One may again rotate the theta parameter into the fermion mass term. The resulting shift in the vacuum energy density is

$$\mathcal{E}^{(d)} = \frac{m_d}{g_d^2} \left[\langle \text{tr } \lambda^1 \lambda^2 \rangle e^{i\theta_d/N} + \langle \text{tr } \bar{\lambda}^2 \bar{\lambda}^1 \rangle e^{-i\theta_d/N} \right] + O(m_d^2), \quad (2.6)$$

where the phase of $\langle \text{tr } \lambda^1 \lambda^2 \rangle$ is now an integer multiple of $2\pi/N$, with the particular value depending on which of the N vacua of the massless daughter theory is under consideration. Changing $\theta_d \rightarrow \theta_d + 2\pi$ is again the same as moving from one vacuum state to the next.

2.1 Parameter mapping between parent and daughter

The parameters of the daughter theory are *not* determined by equating the daughter theory action with the parent theory action, after replacing fields with their projected forms. Rather, as shown in Refs. [5, 6], one must equate the parent action, after projecting fields, with *twice* the daughter action,

$$\mathcal{L}^{(p)} \rightarrow 2 \mathcal{L}^{(d)}. \quad (2.7)$$

(More generally, for a \mathbb{Z}_k orbifold projection the parent action, after projecting fields, must be identified with k times the daughter action.) This implies that the correct relations between parameters of the parent and daughter theories are:

$$g_p^2 = \frac{1}{2} g_d^2, \quad \theta_p = 2 \theta_d, \quad m_p = m_d. \quad (2.8)$$

Both the gauge coupling constant and the vacuum angle of the daughter theory differ from their parent theory counterparts. The relation (2.8) for the gauge couplings is equivalent to equality of 't Hooft couplings in parent and daughter theories; the necessity of this condition for large N orbifold equivalence may be seen in perturbation theory [1]. Equality of fermion masses is obvious, and may also be seen in perturbation theory. The relation (2.8) between parent and daughter theta angles cannot be seen in perturbation theory but, as noted in footnote 2, this relation also follows from the mapping of the holomorphic coupling in cases where both the parent and daughter theories are supersymmetric.

2.2 Energy densities and topological susceptibilities

The relation (2.7) connecting parent and daughter theory actions also applies to their Hamiltonians [6]. Non-perturbative large N equivalence between the parent and daughter theories will be valid provided the \mathbb{Z}_2 theory space symmetry is not spontaneously broken in the daughter.⁵ In this case, their vacuum energies must satisfy the same relation at large N ,

$$\mathcal{E}^{(\text{p})} = 2 \mathcal{E}^{(\text{d})} \times [1 + O(1/N^2)]. \quad (2.9)$$

The factor of two in this relation should not be surprising. One way to understand it physically is to consider the free energy at temperature T , instead of the vacuum energy. At asymptotically high temperatures, the free energy density effectively counts the number of degrees of freedom, half of which are removed by the orbifold projection.⁶

Using the relations (2.8) for the coupling constants, plus (2.4) for the fermion bilinear, one may immediately see that the expressions (2.2) and (2.6) for the first order vacuum energy shift due to a fermion mass are completely consistent with the relation (2.9) between vacuum energy densities.⁷ The coinciding theta dependence of the vacuum energy, given the relation (2.8) between θ -angles, also implies that the topological susceptibilities of the two

⁵We assume that the $\lambda \rightarrow -\lambda$ [or $(-1)^F$] symmetry in the parent, used to define the orbifold projection, cannot be spontaneously broken. This is automatic provided Lorentz invariance is unbroken. Since gauge symmetries can never be spontaneously broken [18], it is only \mathbb{Z}_2 symmetry breaking in the daughter which could prevent large N equivalence between parent and daughter.

⁶Explicitly, the free energy density in the parent is $\mathcal{F}^{(\text{p})} = -\frac{\pi^2}{24} (2N)^2 T^4 [1 + O(2Ng_p^2)]$, while in the daughter $\mathcal{F}^{(\text{d})} = -\frac{\pi^2}{24} (2N^2) T^4 [1 + O(Ng_d^2)]$. Hence $\mathcal{F}^{(\text{p})} = 2 \mathcal{F}^{(\text{d})}$ at high temperature.

⁷If fermion mass is sufficiently large, then both parent and daughter theories have unique ground states, and large N parent-daughter equivalence is guaranteed to hold throughout the large-mass phase. The only way the equivalence could fail in the massless limit is if the daughter theory has a \mathbb{Z}_2 symmetry breaking phase transition at some non-zero mass m_* . This critical mass would have to be of order of the strong scale of the theory, $m_* = C \Lambda$. But as we discuss in the remainder of this section, available information regarding the dynamics of the daughter theory is consistent with the expectation that $C = 0$ and the absence of any \mathbb{Z}_2 symmetry broken phase.

theories satisfy the correct relation for connected correlation functions under large N orbifold equivalence [6]. For a \mathbb{Z}_2 projection, this is⁸

$$\begin{aligned} & \lim_{N \rightarrow \infty} \int d^4x \left\langle \frac{1}{16\pi^2} \text{tr} F \tilde{F}(x) \frac{1}{16\pi^2} \text{tr} F \tilde{F}(0) \right\rangle_{\text{conn}}^{(p)} \\ &= \lim_{N \rightarrow \infty} \frac{1}{2} \int d^4x \left\langle \sum_{j=1}^2 \frac{1}{16\pi^2} \text{tr} F^j \tilde{F}^j(x) \sum_{l=1}^2 \frac{1}{16\pi^2} \text{tr} F^l \tilde{F}^l(0) \right\rangle_{\text{conn}}^{(d)}. \end{aligned} \quad (2.11)$$

With the correct mapping between parent and daughter theories, there is no sign of any failure of large N orbifold equivalence in this example of a \mathbb{Z}_2 projection of $\mathcal{N}=1$ supersymmetric Yang-Mills theory, in infinite volume. The apparent mismatch found by the authors of Refs. [7, 9] was due to an inappropriate assumption of coinciding vacuum energies and/or theta angles.

2.3 Anomalies

Gauge and gravitational contributions to the chiral anomaly provide an instructive test of the consistency of the mapping between parent and daughter theories. The chiral currents in the parent and daughter theories are given by

$$J_p^\mu \equiv \frac{1}{g_p^2} \text{tr} \bar{\lambda} \bar{\sigma}^\mu \lambda, \quad (2.12)$$

and

$$J_d^\mu \equiv \frac{1}{g_d^2} \text{tr} (\bar{\lambda}^1 \bar{\sigma}^\mu \lambda^1 + \bar{\lambda}^2 \bar{\sigma}^\mu \lambda^2). \quad (2.13)$$

One may repackage the two bifundamental Weyl fermions of the daughter theory into a single Dirac fermion $\Psi = \begin{pmatrix} \lambda^1 \\ \lambda^2 \end{pmatrix}$, in which case the daughter chiral current acquires the familiar form $J_d^\mu = \frac{1}{g_d^2} \text{tr} \bar{\Psi} \gamma^\mu \gamma^5 \Psi$. The currents (2.12) and (2.13) are normalized to give fermions equal charge assignments in the two theories, so that the corresponding chiral charges $Q = \int d^3x J^0(x)$ generate the same phase rotation when acting on the fermion fields in either theory. Under the orbifold projection, the chiral current J_p^μ is *not* mapped to J_d^μ , but rather

$$J_p^\mu \rightarrow 2 J_d^\mu. \quad (2.14)$$

⁸For a \mathbb{Z}_k projection connecting $U(kN)$ and $U(N)^k$ gauge theories, connected L -point correlators of neutral single-trace operators, normalized to have finite non-trivial large N expectation values, are related by

$$\lim_{N \rightarrow \infty} (kN)^{2(L-1)} \langle \mathcal{O}_p(x_1) \cdots \mathcal{O}_p(x_L) \rangle_{\text{conn}}^{(p)} = \lim_{N \rightarrow \infty} (kN^2)^{L-1} \langle \mathcal{O}_d(x_1) \cdots \mathcal{O}_d(x_L) \rangle_{\text{conn}}^{(d)}, \quad (2.10)$$

where the projection takes $\mathcal{O}_p \rightarrow \mathcal{O}_d$ [6]. For the topological susceptibility, $\mathcal{O}_p = \frac{1}{kN} \text{tr} F \tilde{F} / (16\pi^2)$ and $\mathcal{O}_d = \frac{1}{kN} \sum_{j=1}^k \text{tr} F^j \tilde{F}^j / (16\pi^2)$. One way to understand the different factors on either side of relation (2.10) is by analogy with the usual $\hbar \rightarrow 0$ limit. L -point connected correlators are proportional to \hbar^{L-1} . In the large N limit under discussion, the dimension of the gauge group (or the number of gluons) plays the role of \hbar , namely, $(kN)^2$ for the parent theory and kN^2 for the daughter. Therefore, when comparing connected correlators between the two theories, one must scale the parent correlators by a factor of k^{L-1} , relative to the daughter, in order to compensate for the differing effective values of \hbar .

This is consistent with the definitions (2.12) and (2.13), combined with the relation (2.8) between gauge couplings. It may also be seen as a consequence of the fact that generators of chiral symmetry transformations must obey the same relation under orbifold projection, namely $Q_p \rightarrow 2Q_d$, as do the Hamiltonians of the two theories, which generate time translations, or all the rest of the generators of the infinite dimensional group which creates large N coherent states [6].

The divergence of the chiral current in the parent theory has a gauge field contribution of $\frac{2N}{8\pi^2} \text{tr} F_{\mu\nu} \tilde{F}^{\mu\nu}$ and, in curved space, a gravitational contribution of $-\frac{N^2}{96\pi^2} R_{\mu\nu\alpha\beta} \tilde{R}^{\mu\nu\alpha\beta}$ [19]. (This is $2N^2$ times the contribution of a single Dirac fermion.) The projection takes $\text{tr} F \tilde{F}$ to $\text{tr}(F^1 \tilde{F}^1 + F^2 \tilde{F}^2)$, so applying the orbifold projection to both sides of the chiral anomaly equation in the parent theory gives

$$\nabla_\mu J_p^\mu = -\frac{N^2}{96\pi^2} R_{\mu\nu\alpha\beta} \tilde{R}^{\mu\nu\alpha\beta} + \frac{2N}{8\pi^2} \text{tr} F_{\mu\nu} \tilde{F}^{\mu\nu} \quad (2.15)$$

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ 2\nabla_\mu J_d^\mu = -\frac{N^2}{96\pi^2} R_{\mu\nu\alpha\beta} \tilde{R}^{\mu\nu\alpha\beta} + \frac{2N}{8\pi^2} \text{tr}(F^1 \tilde{F}^1 + F^2 \tilde{F}^2). \end{array} \quad (2.16)$$

Dividing both sides of Eq. (2.16) by two gives the correct chiral anomaly equation for the daughter theory, namely $\nabla_\mu J_d^\mu = -\frac{N^2}{192\pi^2} R_{\mu\nu\alpha\beta} \tilde{R}^{\mu\nu\alpha\beta} + \frac{N}{8\pi^2} \text{tr}(F^1 \tilde{F}^1 + F^2 \tilde{F}^2)$. The coefficient of $R\tilde{R}$ is half as large in the daughter as in the parent, because there are half as many fermionic degrees of freedom in the daughter theory.

One may compare the trace anomalies of the two theories in a completely analogous fashion. The stress-energy tensors are related by

$$T_p^{\mu\nu} \rightarrow 2T_d^{\mu\nu}, \quad (2.17)$$

with the factor of two appearing for the same reason as for the chiral currents or the Hamiltonian. The parent theory trace anomaly (in flat space, to lowest order), and its projection to the daughter, is

$$(T_p)_\mu^\mu = \frac{6N}{16\pi^2} \text{tr} F_{\mu\nu} F^{\mu\nu} \quad (2.18)$$

$$\begin{array}{ccc} \downarrow & \downarrow & \\ 2(T_d)_\mu^\mu = \frac{6N}{16\pi^2} \text{tr}(F^1 F^1 + F^2 F^2). \end{array} \quad (2.19)$$

Dividing both sides of Eq. (2.19) by two gives the correct trace anomaly equation for the daughter theory.

Finally, instead of considering the operator relations (2.15) and (2.16), one may alternatively examine the consistency of the gravitational contribution to the chiral anomaly by considering the three-point correlation function $\partial_\mu \langle J^\mu T^{\alpha\beta} T^{\gamma\delta} \rangle$, since a linearized metric perturbation couples, in either theory, to the stress-energy tensor via a term $h_{\mu\nu} T^{\mu\nu}$. Using the

appropriate mappings (2.14) and (2.17) for these operators, plus the correct relation between connected correlators, as described in footnote 8, one sees that the $\langle JTT \rangle$ correlator in the parent maps to twice the $\langle JTT \rangle$ correlator in the daughter — correctly reflecting the fact that the coefficient of $R\tilde{R}$ in the chiral anomaly of the daughter theory is half as large as in the parent.

It should be emphasized that the consistency of the above relations among anomalies does not constitute a test of the validity of non-perturbative equivalence between parent and daughter theories, since chiral and trace anomalies depend only on short distance perturbative physics (and not on the \mathbb{Z}_2 symmetry realization in the daughter theory). The fact that anomaly relations of the parent theory are mapped onto the correct anomaly relations in the daughter is a logical consequence of the perturbative equivalence between planar diagrams [1], and merely serves to illustrate how the correct mappings of gauge couplings, operators, and connected correlators all fit together in a consistent fashion.

2.4 Theta dependence and vacuum structure

In this example of a \mathbb{Z}_2 projection of $\mathcal{N}=1$ supersymmetric Yang-Mills theory, the parent theory (with no gluino mass) has $2N$ distinct degenerate vacua, while the daughter theory has only N . This would appear to be *prima-facie* evidence against a non-perturbative large N equivalence. However, large N orbifold equivalence only applies to physical quantities which can be extracted from the leading large N behavior of the free energy density, or from connected correlators of neutral single-trace gauge invariant operators (*i.e.*, invariant under the discrete projection symmetry in the parent, and the discrete theory space symmetry in the daughter) [5, 6]. As noted in the introduction, this includes mass spectra and scattering amplitudes of particles in the relevant neutral symmetry channels, but this information does not directly yield a count of the number of vacua.⁹

Despite the difference in the number of vacua, large N orbifold equivalence *is* applicable to all vacua of the parent theory, not just some subset of “corresponding” vacua (as suggested in Ref. [3]). The resolution of this apparent paradox lies in the rescaling (2.8) between the parent and daughter theta parameters. The parent theta parameter θ_p is an angle which is only defined modulo 2π . Because of this, the relation (2.8) between parent and daughter

⁹At low, but non-zero, temperature, the number of degenerate zero-temperature vacua appears as an overall multiplicative factor in the partition function. Hence, the number of vacua makes an $O(\ln N)$ contribution to the free energy density $\mathcal{F} \equiv -(\ln Z)/(\beta\mathcal{V})$, and the mis-match in the number of vacua contributes $\ln 2$ to the difference in $\beta\mathcal{V}\mathcal{F}$ between the two theories. However, the differing spectra of particles in the non-neutral symmetry channels *also* produces $O(1)$ differences in the free energy density. Large N orbifold equivalence, for the free energy, only applies to the leading $O(N^2)$ contribution (which happens to be temperature independent in the low-temperature confining phase).

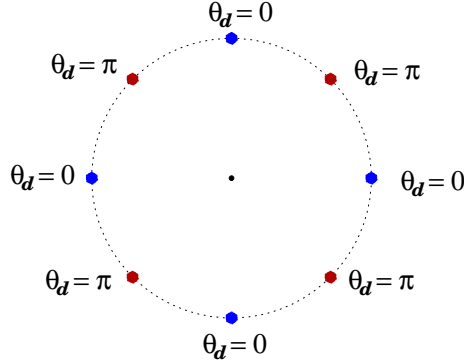


Figure 1: The connection between the choice of vacua in the parent theory, plotted in the complex plane of $\langle\lambda\lambda\rangle^{(p)}$, and the choice of theta parameter for the associated daughter theory, illustrated for $N = 4$ and $\theta_p = 0$.

theta parameters is necessarily double-valued,

$$\theta_d = \frac{1}{2} \theta_p \quad \text{or} \quad \theta_d = \frac{1}{2} \theta_p + \pi, \quad (2.20)$$

so the mapping between parent and daughter theories is really a one-to-two mapping.

In the massless theory, the choice of the daughter theta angle, for a given parent theta angle, depends on which vacua in the parent theory one chooses to examine. At, for example, $\theta_p = 0$, the N parent vacua for which the phase of $\langle\lambda\lambda\rangle$ is an even multiple of π/N map to the N vacua in the daughter theory with $\theta_d = 0$, while the other N parent vacua, where $\langle\lambda\lambda\rangle$ has a phase that is an odd multiple of π/N , map to the N vacua of the daughter theory with $\theta_d = \pi$. This is illustrated in Fig. 1.

With a small fermion mass turned on, the vacuum degeneracy of the massless theories is lifted and the energy densities of the different vacua acquire theta dependence as shown in Eqs. (2.2) and (2.6), and illustrated in Fig. 2. After rescaling θ_p by $1/2$, half of the curves in the parent theory graph coincide with the daughter theory curves, while the other half of the parent curves coincide with the daughter theory curves shifted left (or right) by π . The true ground state of the parent theory changes discontinuously and the ground state energy density has a cusp when $\theta_p = \pi \pmod{2\pi}$. At a given value of θ_p , properties in the true ground state of the parent theory map to corresponding properties in the ground state of the daughter theory with θ_d equal to either $\frac{1}{2} \theta_p$, or $\pi + \frac{1}{2} \theta_p$, depending on which value yields the lower vacuum energy. Physical quantities in both parent and daughter theories have theta dependence which is periodic with period 2π . This would be inconsistent with the $\theta_p = 2\theta_d$ relation, were it not for the two-to-one nature of the mapping between daughter and parent theories.

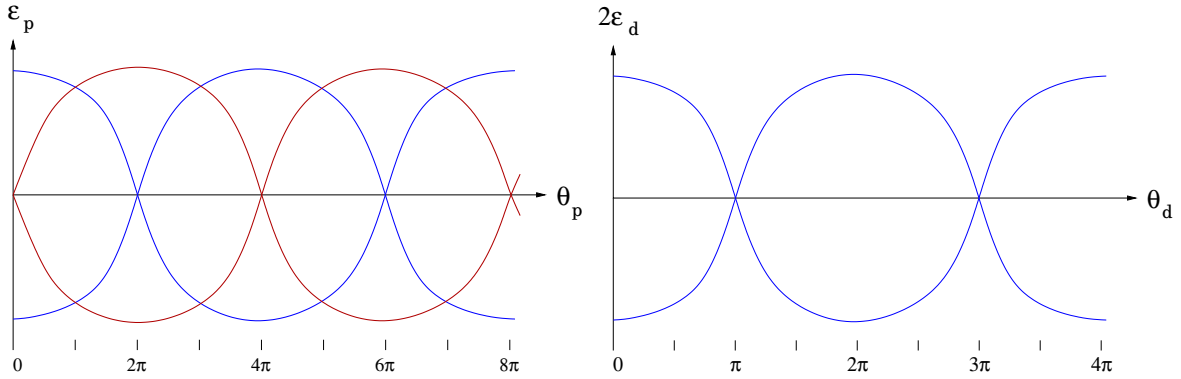


Figure 2: Theta dependence of vacuum energy densities in the parent (left) and daughter (right) theories, illustrated for $N = 2$, when a small fermion mass is turned on. If the daughter theory graph is superimposed on the same graph shifted left (or right) by π , then the result coincides with the parent theory graph after rescaling the theta axis by a factor of two, illustrating the connection between the choice of vacua in the parent and the choice of θ in the daughter.

2.5 Domain walls

Any theory which spontaneously breaks a discrete symmetry will have stable domain walls which interpolate between different degenerate vacua. In the $U(2N)$ supersymmetric parent theory, there are stable BPS domain walls connecting any pair of vacua. The tension of domain walls connecting vacua whose fermion condensate phases differ by $\Delta k (\pi/N)$ is

$$T_p(\Delta k) = C_p N^2 \sin\left(\frac{\pi \Delta k}{2N}\right), \quad \Delta k = 1, \dots, 2N-1, \quad (2.21)$$

where C_p is a pure number times the strong scale Λ^3 [20]. For large N , the tension of near-minimal domain walls [for which Δk is fixed as N grows] is $O(N)$, while the tension of near-maximal domain walls [with $\Delta k/N$ held fixed as N grows] is $O(N^2)$. Domain walls with $\Delta k > 1$ are stable. The binding energy density relative to Δk separate minimal domain walls, $T_p(\Delta k) - \Delta k T_p(1)$, is $O(1/N)$ for near-minimal walls and vanishes as $N \rightarrow \infty$, while the binding energy of near-maximal walls is $O(N^2)$.

The domain wall tension should be viewed as the change in the ground state energy, per unit area, produced by changing from ordinary periodic boundary conditions to “twisted” periodic boundary conditions for which fermion fields at, say, $z = -\infty$ are identified with the fields at $z = +\infty$ only after applying a discrete chiral rotation which changes the phase of $\text{tr } \lambda \lambda$ by a chosen multiple of π/N . Therefore, the appropriate mapping of domain wall tension between parent and daughter theories is the same as for ground state energy densities. For our \mathbb{Z}_2 projection, this means the tension of a domain wall in the parent theory should,

for large N , map to twice the tension of a corresponding domain wall in the daughter theory,

$$T_p = 2T_d \times [1 + O(1/N^2)]. \quad (2.22)$$

There is an obvious puzzle with this correspondence: the daughter theory (for a given value of θ_d) has half as many vacua as the parent, and consequently has fewer domain walls. Choose, for convenience, $\theta_p = 0$. Even-even domain walls — which connect two vacua whose condensate phases are both even multiples of π/N — have an obvious mapping to domain walls in the daughter theory with $\theta_d = 0$. Similarly, odd-odd domain walls are mapped to domain walls in the daughter theory with $\theta_d = \pi$. If large N equivalence holds for these domain walls, then the large N behavior of the tension $T_d(\Delta l)$ of domain walls in the daughter theory [connecting vacua whose condensate phases differ by $\Delta l (2\pi/N)$] must be

$$T_d(\Delta l) = C_d N^2 \sin\left(\frac{\pi \Delta l}{N}\right) \times [1 + O(1/N^2)], \quad \Delta l = 1, \dots, N-1, \quad (2.23)$$

with $C_d = \frac{1}{2} C_p$, so that $T_p(2\Delta l) = 2T_d(\Delta l)$ as $N \rightarrow \infty$.

But what about even-odd domain walls in the parent theory (*i.e.*, walls for which Δk is odd)? These domain walls have no analogs in the daughter theories with either value of θ_d . Does a lack of corresponding domain walls in the daughter theory imply a failure of large N equivalence between the parent and daughter theories (which can only occur if the \mathbb{Z}_2 theory space symmetry is spontaneously broken in the daughter)?

No. The resolution of this apparent problem lies in the divergence of the domain wall tension as $N \rightarrow \infty$. At finite N , any domain wall will have fluctuations (quantum and thermal) in its position. But the mean square size of such fluctuations is inversely proportional to the domain wall tension. The diverging tension in the large N limit means that domain walls, in this limit, are effectively *non-dynamical*.

Consider, for simplicity, the minimal domain wall in the parent theory, connecting the $k = 0$ and $k = 1$ vacuum states (so the phase of the condensate varies from 0 to π/N as one crosses the wall), with $\theta_p = 0$. We have already argued that properties in the $k = 0$ vacuum state of the parent theory map to properties in the corresponding vacuum state in the daughter theory with $\theta_d = 0$, while properties in the $k = 1$ parent vacuum map to properties in its corresponding vacuum state in the daughter theory with $\theta_d = \pi$. Therefore, it seems clear that properties of a non-dynamical (as $N \rightarrow \infty$) domain wall interpolating between the $k = 0$ and $k = 1$ parent vacua should map onto a daughter theory in which $\theta_d = 0$ on one side of the non-dynamical wall, and $\theta_d = \pi$ on the other side, as illustrated in Fig. 3. In other words, the multivaluedness of the mapping of θ parameters between parent and daughter

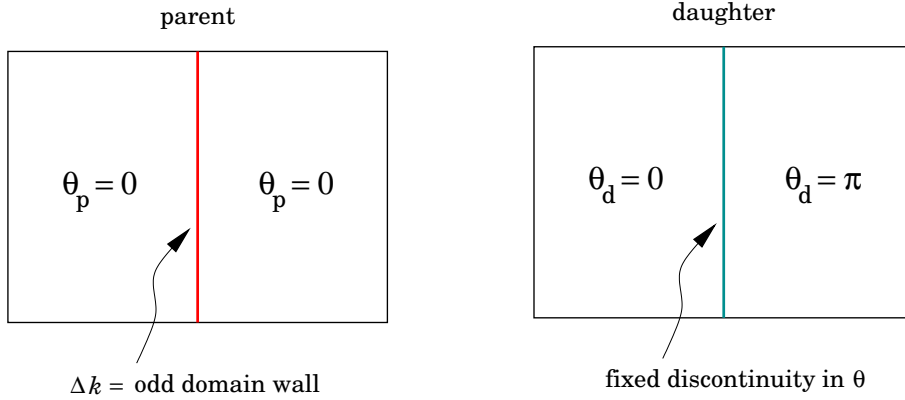


Figure 3: Mapping of parent theory domain walls with Δk odd onto a daughter theory whose theta parameter has a discontinuity of π across a flat interface.

theories can become a discontinuous non-translationally invariant mapping $\theta_p \rightarrow \theta_d(x)$ when considering domain walls, which are non-translationally invariant static equilibrium states.¹⁰

In summary, there is a natural mapping of every domain wall in the parent theory to a corresponding wall in an appropriate daughter theory, provided one considers daughter theories where $\theta_d = \frac{1}{2} \theta_p$, $\theta_d = \frac{1}{2} \theta_p + \pi$, or $\theta_d = \frac{1}{2} \theta_p + \pi n(x)$, where $n(x)$ is a unit step function across a flat interface. Since the daughter theory is not supersymmetric, no exact evaluation of the resulting domain wall tensions is available. But there is no apparent inconsistency with the prediction of large N orbifold equivalence, which requires that relation (2.22) be satisfied for all pairs of corresponding domain walls.

3. \mathbb{Z}_k projections with $k > 2$

One may contemplate starting with $U(kN)$ supersymmetric Yang-Mills theory and applying a \mathbb{Z}_k projection with $k > 2$. Such projections were discussed in Refs. [2, 3, 7, 9]. If one chooses a projection group which is generated by the \mathbb{Z}_k gauge transformation $\gamma \equiv \text{diag}(\omega^0, \omega^1, \omega^2, \dots, \omega^{k-1})$, with $\omega \equiv e^{2\pi i/k}$ times an $N \times N$ identity matrix, then the daughter theory consists of k decoupled copies of $U(N)$ supersymmetric Yang-Mills theory. If one

¹⁰In a box with twisted periodic boundary conditions, as described above, the divergence of the domain wall energy as $N \rightarrow \infty$ implies that translation invariance is spontaneously broken in this limit, reflecting the continuous degeneracy associated with translating the domain wall in the normal direction. At finite N (and in a box with finite transverse size), this overall translation mode of a domain wall would be quantized, producing a zero momentum ground state and excited eigenstates with non-zero momentum P_z (for a domain wall parallel to the x - y plane), and excitation energies equal to $P_z^2/(2M)$ where M is the domain wall tension times its area. When the tension diverges as $N \rightarrow \infty$, this band of eigenstates becomes an infinitely degenerate ground state, signaling the spontaneous breaking of translation symmetry.

instead chooses a projection group which is generated by the product of the gauge transformation γ times a discrete chiral rotation of the fermion λ by $e^{2\pi i/k}$, then the result is a $U(N)^k$ gauge theory with bifundamental Weyl fermions transforming under “nearest neighbor” gauge groups, whose theory space graph is illustrated in Fig. 4.

The daughter theory in this latter case is a chiral gauge theory (when $k > 2$); one cannot add gauge invariant mass terms for the fermions. But the parent supersymmetric Yang-Mills theory is a non-chiral vectorlike theory. It would be truly remarkable if a large N equivalence related physical properties of these two theories. Alas (but not surprisingly), this is not the case. The discrete chiral symmetry transformation used to define this orbifold projection is part of the non-anomalous \mathbb{Z}_{2kN} chiral symmetry of the parent theory. This chiral symmetry is known to break spontaneously down to \mathbb{Z}_2 . Therefore, any projection which involves chiral symmetry transformations other than the \mathbb{Z}_2 symmetry of $(-1)^F$ will violate the condition of unbroken projection symmetry in the parent necessary for large N orbifold equivalence.

The first choice of projection, generating k decoupled copies of $U(N)$ supersymmetric Yang-Mills theory from a single $U(kN)$ theory, is a completely valid, but perhaps unremarkable, large N equivalence. Since there are no interactions between the multiple copies of $U(N)$ theories in the daughter, the permutation symmetry among the different copies cannot possibly be spontaneously broken. Neither can the projection symmetry in the parent theory, since gauge symmetries cannot be spontaneously broken [18]. All gauge invariant operators in the parent are neutral under this projection. The mapping relates parent and daughter theories with coinciding values of ’t Hooft couplings and theta parameters rescaled by k , that is $g_p^2 = \frac{1}{k} g_d^2$ and $\theta_p = k \theta_d$.

This large N equivalence relating expectation values in the parent and daughter theories (or multi-point connected correlators when scaled by appropriate powers of $(kN)^2$ in the parent and kN^2 in the daughter [6]) merely reflects the existence of the large N limit in

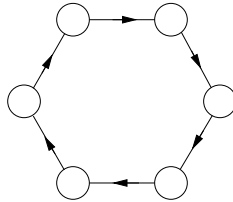


Figure 4: The theory space for a \mathbb{Z}_k projection of $\mathcal{N}=1$ $U(kN)$ supersymmetric Yang-Mills theory yielding a non-supersymmetric $U(N)^k$ daughter theory with bifundamental fermions, for $k=6$. Nodes represent individual $U(N)$ gauge group factors, while the arrows represent bifundamental fermions transforming in the fundamental and antifundamental representation, respectively, of the gauge group factors at either end.

the original super-Yang-Mills theory. One noteworthy feature is the mapping of discrete vacua. The parent has kN vacua while the daughter has N^k , of which N are invariant under theory space permutation symmetry. The difference between the number of parent vacua and permutation symmetric daughter vacua is once again compensated by the k -fold multivaluedness of the theta parameter mapping, $\theta_d = (\theta_p + 2\pi j)/k$, $j = 0, 1, \dots, k-1$. As a result, observables (such as $\langle \lambda\lambda \rangle$) in any vacuum state of the parent map to corresponding observables in one of the permutation symmetric ground states of the daughter theory for one of the choices of θ_d .

The above discussion of domain walls also applies directly to this \mathbb{Z}_k projection. Domain walls in the parent theory which connect vacua separated by multiples of k map onto domain walls in the daughter theory with a single value of theta, while domain walls connecting vacua which are not separated by multiples of k map to daughter theories in which θ is discontinuous (by a multiple of $2\pi/k$) across a flat interface.

4. Multiple fermion flavors

One may add additional adjoint representation fermions to the $U(2N)$ supersymmetric Yang-Mills parent theory, to give a non-supersymmetric $U(2N)$ gauge theory with $n_f > 1$ adjoint Weyl fermions.¹¹ This theory has a non-anomalous $SU(n_f) \times \mathbb{Z}_{4Nn_f}$ flavor symmetry,¹² which is expected to break spontaneously down to $SO(n_f)$, due to the formation of a bilinear fermion condensate, giving rise to $2N$ disjoint connected components in the vacuum manifold, each of which is the coset space $SU(n_f)/SO(n_f)$. The different components in the vacuum manifold are distinguished by the phase of the determinant of the fermion condensate, in complete analogy to the phase of the condensate in $\mathcal{N}=1$ supersymmetric Yang-Mills theory. The breaking of the continuous chiral symmetry results in $\frac{1}{2}n_f(n_f+1) - 1$ Goldstone bosons.

The same \mathbb{Z}_2 projection discussed in section 2, applied to this multi-flavor theory, yields a $U(N) \times U(N)$ daughter gauge theory with n_f flavors of bifundamental Dirac fermions. This daughter theory has a $SU(n_f)_L \times SU(n_f)_R \times \mathbb{Z}_{2Nn_f}$ flavor symmetry¹³ which is expected to break spontaneously down to the diagonal $SU(n_f) \times \mathbb{Z}_2$ [where the \mathbb{Z}_2 is $(-1)^F$], resulting in

¹¹The leading term of the beta function is proportional to $(\frac{11}{3} - \frac{2}{3}n_f)N$, so preserving asymptotic freedom, which our discussion assumes, restricts the number of flavors to at most five.

¹²More precisely, one should mod out from the $SU(n_f) \times \mathbb{Z}_{4Nn_f}$ product a \mathbb{Z}_{n_f} subgroup which is common to both factors. And the unbroken symmetry is $O(n_f)$ rather than $SO(n_f)$ if n_f is odd. In either case, the center of the unbroken flavor symmetry group is the single \mathbb{Z}_2 generated by $(-1)^F$, which is relevant for understanding the pattern of discrete chiral symmetry breaking.

¹³A double-counted $\mathbb{Z}_{n_f} \times \mathbb{Z}_{n_f} \times \mathbb{Z}_2$ factor should similarly be divided out of this product symmetry group. There is also an overall $U(1)_B$ symmetry, but in a $U(N) \times U(N)$ theory with bifundamental fermions this is just part of the global gauge symmetry.

N disjoint connected components in the vacuum manifold, which are again distinguished by the phase of the determinant of the fermionic condensate. The breaking of the continuous chiral symmetry leads in this case to $n_f^2 - 1$ Goldstone bosons.

This mis-match in the number of Goldstone bosons in the multi-flavor parent and daughter theories was previously noted in Ref. [7], where it was asserted to be evidence for failure of large N orbifold equivalence in these multi-flavor theories. However, this is a mis-interpretation.

The orbifold equivalence only applies to neutral symmetry channels (those which are invariant under the \mathbb{Z}_2 projection symmetry in the parent, and the \mathbb{Z}_2 theory space symmetry interchanging gauge group factors in the daughter). In the parent, all Goldstone bosons are neutral, but in the daughter one may easily show that $\frac{1}{2}n_f(n_f-1)$ Goldstone bosons are odd under the theory space \mathbb{Z}_2 symmetry, while $\frac{1}{2}n_f(n_f+1) - 1$ of them are even — correctly matching the number in the parent theory.

The parent and daughter theories under consideration have both discrete and continuous symmetry breaking. The latter gives rise to Goldstone bosons, whereas the former is responsible for the emergence of multiple connected components in the vacuum manifold and hence domain walls which interpolate between different components. The number of distinct connected components is $2N$ in the parent theory and N in the daughter. This situation, in essence, is identical to the case of $\mathcal{N} = 1$ supersymmetric Yang-Mills theory (which is just the special case of $n_f = 1$) and its \mathbb{Z}_2 projection. This difference in the number of connected components is, once again, fully consistent with non-perturbative large N equivalence due to the double valuedness of the mapping from parent to daughter theta angles, as discussed in section 2.

5. \mathbb{Z}_2 projection of $\mathcal{N} = 4$ supersymmetric Yang-Mills

Numerous authors [11–16] have examined the relation between $U(2N)$ $\mathcal{N} = 4$ supersymmetric Yang-Mills theory and its \mathbb{Z}_2 projection which yields a non-supersymmetric $U(N)^2$ theory with bifundamental fermions and adjoint scalars, in both the weak-coupling limit using perturbative methods, and in the strong-coupling limit using AdS/CFT duality. It is natural to ask what lessons, if any, can be extracted from the $\mathcal{N} = 4$ case that are relevant to the $\mathcal{N} = 1$ case discussed in this paper.

The validity of large N equivalence between parent and daughter theories reduces, in either case, to the question of the realization of the \mathbb{Z}_2 symmetry interchanging gauge group factors in the non-supersymmetric daughter theory. In the case of the $\mathcal{N} = 4$ parent, there is compelling evidence that this \mathbb{Z}_2 symmetry *is* spontaneously broken in the daughter theory.

At weak coupling, the one-loop effective potential for the adjoint scalars has a Coleman-Weinberg form, with non-trivial minima in which the twisted operator $\text{tr}[(\Phi_a^1)^2 - (\Phi_a^2)^2]$ has a non-zero expectation value [11, 12]. At strong coupling, a dual description of the daughter theory in terms of type 0 string theory was conjectured [13]. This theory has a tachyon which couples to the twisted operator $\mathcal{O} \equiv \frac{1}{2}\text{tr}[F^1 F^1 - F^2 F^2] + \text{tr}[(D\Phi_a^1)^2 - (D\Phi_a^2)^2] + \dots$ [15], thus suggesting that tachyon condensation signals spontaneous breaking of the \mathbb{Z}_2 symmetry. If one ignores this instability (which must lead to a deformation of the geometry), one finds an unphysical complex anomalous dimension for the operator \mathcal{O} at strong coupling (or small curvature). Hence, for this \mathbb{Z}_2 projection of $\mathcal{N} = 4$ super-Yang-Mills theory, it seems clear that large N equivalence between parent and daughter theories does not hold due to spontaneous \mathbb{Z}_2 symmetry breaking in the daughter theory.

However, none of the analysis used for the $\mathcal{N} = 4$ case is applicable to the \mathbb{Z}_2 projection of $\mathcal{N} = 1$ super-Yang-Mills. In the latter case, there are no (non-composite) scalar fields and symmetry breaking cannot be studied using weak-coupling perturbative methods. Both parent and daughter theories are asymptotically free, and neither has a known gravitational dual in which a supergravity approximation is valid. As always, the presence or absence of symmetry breaking depends on the detailed dynamics of a theory. So failure of large N equivalence in the case of a \mathbb{Z}_2 projection of $\mathcal{N} = 4$ super-Yang-Mills tells one nothing about the validity of large N equivalence in the $\mathcal{N} = 1$ case.

6. Conclusions

\mathbb{Z}_k projections of $\mathcal{N} = 1$ supersymmetric $U(kN)$ Yang-Mills theory, producing $U(N)^k$ daughter theories, provide very instructive examples of how large N equivalence between theories related by orbifold projection can work, or fail to work. With the proper mapping of observables between parent and daughter theories, large N orbifold equivalence must hold unless the chosen \mathbb{Z}_k symmetry used to define the projection is spontaneously broken in the parent, or the \mathbb{Z}_k symmetry permuting gauge group factors is spontaneously broken in the daughter [6].

For \mathbb{Z}_k projections with $k > 2$ that yield chiral theories, large N equivalence does fail due to chiral symmetry breaking in the parent (and the use of a spontaneously broken chiral transformation in the chosen projection). So in this case, knowledge about properties of the parent theory, in the large N limit, cannot be used to infer properties of the large N daughter theory.

For \mathbb{Z}_2 projections of $\mathcal{N} = 1$ super-Yang-Mills, the projection symmetry cannot be spontaneously broken in the parent, so large N equivalence must be valid unless the \mathbb{Z}_2 symmetry exchanging gauge group factors is spontaneously broken in the daughter theory. Therefore,

if any consistency test of large N equivalence can be shown to fail, this would immediately imply spontaneous symmetry breaking of the \mathbb{Z}_2 theory space symmetry in the daughter.

Contrary to previous claims, a comparison of vacuum structure reveals complete consistency with large N orbifold equivalence for the \mathbb{Z}_2 projection of super-Yang-Mills (on R^4), or its multiflavor generalizations. Properties in any vacuum state of the parent correctly map to properties in a corresponding vacuum state of the daughter, provided one recognizes the multi-valuedness in the mapping of the theta parameter between parent and daughter theories, and provided one uses the correct mappings of parameters, observables, and correlation functions between the two theories.

The correspondence between domain walls in parent and daughter theories is somewhat subtle. “Even” domain walls in the parent naturally correspond with domain walls in the daughter theory at a given value of theta, but “odd” domain walls map to daughter theories in which the theta parameter has a discontinuity across a fixed flat interface. This can be consistent due to the fact that domain walls become non-dynamical, with vanishingly small fluctuations in position, as $N \rightarrow \infty$.

A simple but essential point, which seems not to have been adequately appreciated in some previous literature, is that large N equivalence between the (neutral sectors of) parent and daughter theories does not mean strict equality between all corresponding physical quantities. Rather, it means that there is a well-defined mapping connecting physical quantities in the two theories. For a \mathbb{Z}_k projection, this mapping necessarily involves rescaling by appropriate powers of k . The required rescalings are not arbitrary or difficult to understand. The situation is completely analogous to comparing results in semi-classical quantum theories at two different values of \hbar : for quantities which scale like \hbar^n , such as $n+1$ point connected correlators, one must obviously rescale by $(\hbar_1/\hbar_2)^n$ before comparing.

In summary, we find no sign of any inconsistency which would force one to conclude that large N equivalence cannot hold between the parent $\mathcal{N} = 1$ supersymmetric Yang-Mills theory and its non-supersymmetric \mathbb{Z}_2 projection. Available evidence supports the expectation that the \mathbb{Z}_2 theory space symmetry remains unbroken in the daughter theory (on R^4), and hence that there is a non-perturbative large N equivalence between these theories. Verifying, or disproving, this expectation via lattice simulations would be desirable and should be feasible.

Note added:

The recent preprint [21] discusses the same \mathbb{Z}_2 projection of super-Yang-Mills theory, and examines a number of physical quantities which can be used as consistency tests of large N equivalence between parent and daughter theories. The authors of Ref. [21] argue that all their tests fail, and therefore assert that spontaneous breaking of \mathbb{Z}_2 symmetry must occur

in the daughter theory. However, the apparent inconsistencies noted in Ref. [21] are all consequences of incorrect mappings of observables and/or connected correlators between parent and daughter theories. When the correct mappings are used, these apparent inconsistencies disappear.¹⁴

Acknowledgments

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¹⁴The supposed mis-match in the vacuum structure at $\theta_p = 2\pi$, discussed in section 2 of Ref. [21], is a consequence of overlooking the double-valued nature of the mapping from θ_p to θ_d . With a non-zero mass turned on, it is clear that properties in the unique ground state of the parent theory at $\theta_p = 2\pi$ correspond to properties in the unique ground state of the daughter theory at $\theta_d = 0$, not at $\theta_d = \pi$. The assertion (on pg. 7) that “the partition functions for parent and daughter must coincide at large N ” is also incorrect. Both partition functions diverge exponentially as $N \rightarrow \infty$. The correct statement of equivalence for the partition functions is that the ratio of their free energies must approach two (not one) as $N \rightarrow \infty$, or $\lim_{N \rightarrow \infty} (2N)^{-2} \ln Z_p = \lim_{N \rightarrow \infty} (2N^2)^{-1} \ln Z_d$.

The claimed inconsistency in gravitational contributions to the axial anomaly, discussed in section 3 of Ref. [21], is a result of the use of incorrect mappings between axial currents, stress-energy tensors, and topological charge densities. As discussed in section 2.3 of this paper, when the correct relations are used the chiral anomaly in the parent theory (including both gauge and gravitational contributions) is properly mapped to the chiral anomaly in the daughter theory, as it must be for purely perturbative equivalence to be valid.

The discussion in section 4 of Ref. [21] of $\langle \text{tr}(F^1 F^1 + F^2 F^2) \rangle$ as an order parameter for the massless theory whose non-zero value, if $O(N)$ [not $O(N^2)$], would signal failure of large N equivalence and hence imply \mathbb{Z}_2 symmetry breaking in the daughter theory is fine — except that there is no evidence that this expectation value, suitably renormalized, is non-vanishing at $O(N)$.

The comparison of domain wall tensions in section 5 asserts that $\text{tr} FF$ maps to $\text{tr}(F^1 F^1 + F^2 F^2)/\sqrt{2}$, and then asserts that valid large N equivalence requires that parent and daughter domain wall tensions coincide. Neither is correct; $\text{tr} FF$ maps to $\text{tr}(F^1 F^1 + F^2 F^2)$ with no $\sqrt{2}$, and the tension in the parent theory should be compared with twice the tension of the corresponding wall in the daughter, as explained in section 2.5 above. Once one corrects the mis-understanding regarding coinciding vacuum structure in parent and daughter theories, we fail to see any basis for concluding that domain walls in the daughter theory can split up into “fractional domain walls.” The same mis-understanding regarding the appropriate mappings of $\text{tr}(FF)$ and $\text{tr}(F\tilde{F})$, and of vacuum energies (namely $\mathcal{E}_{(p)} \rightarrow 2\mathcal{E}_{(d)}$) are responsible for the apparent inconsistencies presented in section 6 and the appendix of Ref. [21].

In summary, the assertions in Ref. [21] that “Ample evidence ... establishes ... nonperturbative nonequivalence” and that “this is the first example of spontaneous breaking of a discrete symmetry in a strongly coupled gauge theory ever established analytically in four dimensions” are unfounded. Spontaneous breaking of the \mathbb{Z}_2 theory space symmetry of the daughter theory (in infinite volume) produced by a \mathbb{Z}_2 projection of $\mathcal{N}=1$ super-Yang-Mills remains a logical possibility — but everything known about this theory, so far, is consistent with the absence of such symmetry breaking, and hence with the validity of large N equivalence in this case.

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